

# Normal bundles to Laufer rational curves in local Calabi-Yau threefolds

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## Abstract

We prove a conjecture by F. Ferrari. Let  $X$  be the total space of a nonlinear deformation of a rank 2 holomorphic vector bundle on a smooth rational curve, such that  $X$  has trivial canonical bundle and has sections. Then the normal bundle to such sections is computed in terms of the rank of the Hessian of a suitably defined superpotential at its critical points.

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**Introduction.** In this paper we consider particular embeddings of smooth rational curves in local Calabi-Yau threefolds, called Laufer curves [6, 5]. (We use the definition of the physics community, calling Calabi-Yau a quasi-projective threefold with trivial canonical bundle; the term “local” refers to non-compactness.) These geometries have shown to be very useful to understand several features of string theories and supersymmetric gauge theories.

In particular they are relevant for brane dynamics and geometric transition/large  $N$  dualities. Geometric transition interprets the resummation of the open string sector of an open-closed string theory as a transition in the target space geometry, connecting two different components of a moduli space of Calabi-Yau threefolds. The local Calabi-Yau that we consider represents the open string side of conjectured geometric transitions. In particular, open topological B-type strings in these geometry reduces to matrix models in which the parameters of the complex structure are the coupling constants.

This is directly connected (via F-terms) to the possibility of geometrically engineering supersymmetric gauge theories in Type IIB string theory. Let  $\mathbb{R}^4 \times X$  be the target space of the theory, where  $X$  is a Calabi-Yau threefold, and  $\mathcal{C}$  a rational curve in  $X$ , with normal bundle  $V$  and  $N$  D5 branes wrapped on it. The effective field theory is a  $\mathcal{N} = 1$  supersymmetric gauge theory with gauge group  $U(N)$ . The space of vacua of this gauge theory, given by the critical points of the effective superpotential, is locally described by the versal deformation space of the curve in  $X$ . For a given vacua, there are  $h^0$  massless chiral superfields in the adjoint representation of the gauge group, where  $h^0 := \dim H^0(\mathcal{C}, V)$ . On the other hand, the number of massless chiral multiplets is equal to the corank of the Hessian of the superpotential at this vacuum. This relation led [3] to conjecture the result expressed in our Proposition 2.

For an account of these aspects, see also [2, 7, 1] and references therein.

From a strictly mathematical viewpoint, the problem is the following. Let  $V$  be a rank-2 holomorphic vector bundle on a rational curve  $\mathcal{C}$  such that its total space has trivial canonical bundle, and assume that  $V$  has a global section. We deform  $V$  to a nonlinear fibration  $X$  in such a way that  $X$  still has trivial canonical bundle and the fibration has sections. The normal bundle to such a section of course splits as a direct sum of two line bundles in view of Grothendieck's classification of vector bundles on curves of genus zero [4]. The problem is to compute these line bundles. The solution is obtained in terms of a superpotential  $W$  than one associates with the deformations of  $V$ : the sections of  $X$  are given by the critical points of  $W$ , and the degrees of the above mentioned line bundles are given, in accordance with a conjecture

by Ferrari [3], by the rank of the Hessian of  $W$  at those critical points.

**Definition of  $X$ .** Let  $\mathcal{C} \simeq \mathbb{P}^1$  be a smooth rational curve and  $V \rightarrow \mathcal{C}$  a rank-2 holomorphic vector bundle on  $\mathcal{C}$ , with  $\det V \simeq K_{\mathcal{C}} \simeq \mathcal{O}(-2)$ , so that the total space of the bundle  $V$  has trivial canonical bundle. Then  $V \simeq \mathcal{O}(-n-2) \oplus \mathcal{O}(n)$  for some  $n$ . We consider deformations of  $V$  given in terms of transition functions in the standard atlas  $\mathcal{U} = \{U_0, U_1\}$  of  $\mathbb{P}^1$  as

$$\begin{cases} z' &= 1/z \\ \omega'_1 &= z^{-n}\omega_1 \\ \omega'_2 &= z^{n+2}(\omega_2 + \partial_{\omega_1} B(z, \omega_1)) \end{cases} \quad (1)$$

Note that the complex manifold  $X$  defined as the total space of this fibration has again trivial canonical bundle. The term  $B(z, \omega)$  is a holomorphic function on  $(U_0 \cap U_1) \times \mathbb{C}$  and is called the *geometric potential*. If we expand the function  $B$  in its second variable

$$B(z, \omega_1) = \sum_{d=1}^{\infty} \sigma_d(z) \omega_1^d \quad (2)$$

each coefficient  $\sigma_d$  may be regarded as a cocycle defining an element in the group

$$H^1(\mathbb{P}^1, \mathcal{O}(-2 - dn)) \simeq H^0(\mathbb{P}^1, \mathcal{O}(nd))^*. \quad (3)$$

**The superpotential.** If we consider  $\mathcal{C}$  as embedded in  $X$  as its zero section, and consider the problem of deforming the pair  $(X, \mathcal{C})$ , the space of versal deformations can be conveniently described by a superpotential [5]. In the case at hand the superpotential  $W$  can be defined as the function of  $n+1$  complex variables given by

$$W(x_0, \dots, x_n) = \frac{1}{2\pi i} \oint_{\mathcal{C}_0} B(z, \omega_1(z)) \, dz \quad (4)$$

where  $z$  and  $z'$  are local coordinates on  $U_0$  and  $U_1$ , and the parameters  $x_0, \dots, x_n$  define sections of the line bundle  $\mathcal{O}(n)$  by letting

$$\omega_1(z) = \sum_{i=0}^n x_i z^i, \quad \omega'_1(z') = \sum_{i=0}^n x_i (z')^{n-i}. \quad (5)$$

One should note that the superpotential  $W$  can be obtained by applying to the function  $B$ , regarded as an element in  $H^0(\mathbb{P}^1, \mathcal{O}(nd))^*$ , the dual of the multiplication morphism

$$H^0(\mathbb{P}^1, \mathcal{O}(n))^{\otimes d} \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(nd)) \quad (6)$$

(here one should regard the dual of  $H^0(\mathbb{P}^1, \mathcal{O}(nd))$  as a space of Laurent tails).

The key to the result we want to prove is the relationship occurring between the superpotential  $W$  and the sections of the fibration  $X \rightarrow \mathcal{C}$  (cf. [5, 3]).

**Lemma 1.** *The holomorphic sections of the fibration  $X \rightarrow \mathcal{C}$  are in a one-to-one correspondence with the critical points of the superpotential, i.e., with the solutions of the equations*

$$\frac{\partial W}{\partial x_i} = 0, \quad i = 0, \dots, n. \quad (7)$$

*Proof.* This can be verified by explicit calculations [3] after representing the sections of  $X$  as

$$\begin{aligned} \omega_2(z) &= -\frac{1}{2i\pi} \oint_{C_z} \frac{\partial_\omega B(u, \omega_1(u))}{u - z} du \\ \omega'_2(z') &= \frac{1}{2i\pi} \oint_{C_{z'}} \frac{\partial_\omega B(1/u, \omega_1(1/u))}{u^{n+2}(u - z)} du \end{aligned} \quad (8)$$

where the contour  $C_z$  (resp.  $C_{z'}$ ) encircles the points 0 and  $z$  (resp.  $z'$ ). So (5) and (8) yield a rational curve  $\Sigma \subset X$  for each critical point  $(x_0, \dots, x_n)$  of  $W$ .  $\square$

**Ferrari's Conjecture.** Now we state and prove Ferrari's conjecture.

**Proposition 2.** *The normal bundle to the section  $\Sigma$  of  $X$  determined by a critical point  $(x_0, \dots, x_n)$  of  $W$  is  $\mathcal{O}_\Sigma(-r - 1) \oplus \mathcal{O}_\Sigma(r - 1)$  where  $r$  is the corank of the Hessian of  $W$  at that point.*

To calculate the normal bundle to  $\Sigma$  we first need to linearize the transition functions around the given section. Defining new coordinates  $\delta_i =$

$\omega_i - \omega_i(z)$ ,  $\delta'_i = \omega'_i - \omega'_i(z)$ , we obtain

$$\delta'_2 = z^{n+2} (\delta_2 + h(z)\delta_1 + g(z)) \quad (9)$$

where

$$g(z) = \partial_\omega B(z, \omega_1(z)) , \quad h(z) = \partial_\omega^2 B(z, \omega_1(z)) \quad (10)$$

and at a critical point of  $W$  we have  $g(z) = 0$  using relation (22) in the appendix. Furthermore, again from (22), for  $h(z)$  we have

$$h(z) = \sum_{i \leq j=0}^n \partial_i \partial_j W_d^{(k)} z^{-(i+j)-1} \quad (11)$$

up to terms that can be can be readsorbed by holomorphic change of coordinates (see the Appendix).

Now we need the following. Let us consider an extension of vector bundles on  $\mathbb{P}^1$  of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n-2) \longrightarrow \Phi \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow 0 \quad (12)$$

parametrized by a cocycle  $\sigma \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2n-2))$ . With respect to the two standard charts  $U_0, U_1$  and in the coordinate  $z$  of  $U_0$ ,  $\sigma$  can be written as

$$\sigma(z) = \sum_{k=0}^{2n} \tilde{t}_k z^{-k-1} . \quad (13)$$

Let us define a quadratic form (quadratic superpotential) on the global sections of the line bundle  $\mathcal{O}_{\mathbb{P}^1}(n)$ :

$$H(x_0, \dots, x_n) = \sum_{k=0}^{2n} \tilde{t}_k \sum_{\substack{i,j=0 \\ i+j=k}}^n x_i x_j = \sum_{i,j=0}^n H_{ij} x_i x_j . \quad (14)$$

**Lemma 3.** *The vector bundle  $\Phi$  is  $\mathcal{O}_{\mathbb{P}^1}(r-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-r-1)$ , where  $r$  is the corank of the quadratic form  $H$ .*

*Proof.* By Lemma 1 the sections of the bundle  $\Phi$  correspond to the critical points of  $H$ , *i.e.*, to the solutions of the linear system

$$\sum_{j=0}^n H_{ij} x_j = 0 . \quad (15)$$

The dimension of this space is  $r$ , the corank of  $H$ . The only rank two vector bundle over  $\mathbb{P}^1$  with determinant  $\mathcal{O}_{\mathbb{P}^1}(-2)$  and  $r$  linearly independent holomorphic sections is  $\mathcal{O}_{\mathbb{P}^1}(r-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-r-1)$ .  $\square$

The proof of Proposition 2 is now complete: in fact, by (11) the quadratic form  $H$  corresponds to the Hessian of the superpotential  $W$  at its critical points.

## A Some formulas for the potentials

We group here some formulas that turn out to be useful in checking the computations involved in the results presented in this paper.

**The geometric potential.** The geometric potential (deformation term)  $B(z, \omega_1)$  is holomorphic on  $\mathbb{C}^* \times \mathbb{C}$  and can be cast in the form

$$B(z, \omega) = \sum_{d=0}^{\infty} \sum_{k=0}^{dn} t_d^{(k)} B_d^{(k)}(z, \omega) \quad (16)$$

where

$$B_d^{(k)}(z, \omega) = z^{-k-1} \omega^d \quad k = 0, \dots, dn . \quad (17)$$

The terms with  $k < 0$  or  $k > dn$  can be readsorbed by a holomorphic change of coordinates. For  $l := -k - 1 \geq 0$ , we define  $\tilde{\omega}_2 := \omega_2 + dz^l \omega_1^{d-1}$ , and for  $m := k - dn - 1 \geq 0$ , we define

$$\tilde{\omega}'_2 := \omega'_2 - (z')^m (\omega'_1)^{d-1} . \quad (18)$$

**The superpotential** The superpotential that corresponds to  $B_d^{(k)}$ , given by (4), is

$$W_d^{(k)}(x_0, \dots, x_n) = \sum_{\substack{i_1, \dots, i_d=0 \\ i_1 + \dots + i_d = k}}^n x_{i_1} \dots x_{i_d} . \quad (19)$$

We can obtain simple relations for the derivatives of these polynomials:

$$\begin{aligned} \frac{\partial W_d^{(k)}}{\partial x_j} &= \sum_{\substack{i_1, \dots, i_d=0 \\ i_1 + \dots + i_d = k}}^n d \left( \frac{\partial x_{i_1}}{\partial x_j} x_{i_2} \dots x_{i_d} \right) \\ &= d \sum_{\substack{i_1, \dots, i_{d-1}=0 \\ i_1 + \dots + i_{d-1} = k-j}}^n x_{i_1} \dots x_{i_{d-1}} = d W_{d-1}^{(k-j)} \end{aligned} \quad (20)$$

and in general we have

$$\frac{\partial}{\partial x_{j_1}} \dots \frac{\partial}{\partial x_{j_l}} W_d^{(k)} = d(d-1) \dots (d-l+1) W_{d-l}^{(k-j_1 \dots - j_l)} \quad (21)$$

**Relations between the derivatives of the potentials** Given a section  $\omega_1(z)$ , we have

$$\begin{aligned} \partial_\omega B(z, \omega_1(z)) &= \sum_{j=0}^n \frac{\partial W}{\partial x_j} z^{-j-1} + \text{trivial terms} \\ \partial_\omega^2 B(z, \omega_1(z)) &= \sum_{i \leq j=0}^n \partial_i \partial_j W z^{-(i+j)-1} + \text{trivial terms} \end{aligned} \quad (22)$$

where the “trivial terms” can be readsorbed by a holomorphic change of coordinates. We can obtain these results from (16) and (19). We have

$$\partial_\omega B_d^{(k)}(z, \omega_1(z)) = d \sum_{i_1, \dots, i_{d-1}=0}^n x_{i_1} \dots x_{i_{d-1}} z^{i_1 + \dots + i_{d-1} - k - 1} \quad (23)$$

and the only non-trivial terms are such that  $0 \leq -(i_1 + \dots + i_{d-1} - k) \leq n$ . In the same way, for the second derivatives we have

$$\partial_\omega^2 B_d^{(k)}(z, \omega_1(z)) = d(d-1) \sum_{i_1, \dots, i_{d-2}=0}^n x_{i_1} \dots x_{i_{d-2}} z^{i_1 + \dots + i_{d-2} - k - 1} \quad (24)$$

The relevant terms are those with  $0 \leq -(i_1 + \dots + i_{d-1} - k) \leq 2n$ .

## References

- [1] G. Bonelli, L. Bonora and A. Ricco, “Conifold geometries, topological strings and multi-matrix models”, [arXiv: hep-th/0507224].
- [2] C. Curto, “Matrix model superpotentials and Calabi-Yau spaces: an ADE classification”, PhD thesis [arXiv: math.AG/0505111].
- [3] F. Ferrari, “Planar diagrams and Calabi-Yau spaces”, Adv. Theor. Math. Phys. **7** (2004) 619 [arXiv: hep-th/0309151].
- [4] A. Grothendieck, “Sur la classification des fibrés holomorphes sur la sphère de Riemann”, Amer. J. Math. **79** (1957) 121.
- [5] S. Katz, “Versal deformations and superpotentials for rational curves in smooth threefolds”, [arXiv: math.ag/0010289].
- [6] H. Laufer, “On  $\mathbb{CP}^1$  as an exceptional set”. In: *Recent Developments in Several Complex Variables* (J. Fornaess, ed.), Ann. of Math. Stud. Vol. 100, Princeton Univ. Press, Princeton, NJ 1981, 261–275.
- [7] L. Mazzucato, “Remarks on the analytic structure of supersymmetric effective actions”, [arXiv: hep-th/0508234].
- [8] M. Namba, “On maximal families of compact complex submanifolds of complex manifolds”, Tôhoku Math. J. **24** (1972) 581.